Surds

What you should know

How to multiply algebraic terms involving surds.
How to simplify a fraction with a surd in the denominator, known as ‘rationalising the denominator’.

New idea

When you multiply $3-\sqrt{5}$ by $3+\sqrt{5}$ (the same numbers but with the sign between them changed) you get $(3-\sqrt{5})(3+\sqrt{5})=4$.

Task: Rationalising the denominator

You can use the idea above to simplify fractions with terms like $3-\sqrt{5}$ in the denominator, such as $\dfrac{1+\sqrt{5}}{3-\sqrt{5}}$.

$$\frac{1+\sqrt{5}}{3-\sqrt{5}}=\frac{1+\sqrt{5}}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}}$$
$$=\frac{8+4\sqrt{5}}{4}$$
$$=2+\sqrt{5}$$

$\dfrac{3+\sqrt{5}}{3+\sqrt{5}}$ is a fraction which is equal to 1.

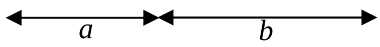
- Why does $(3-\sqrt{5})(3+\sqrt{5})=4$?
- Why does $(1+\sqrt{5})(3+\sqrt{5})=8+4\sqrt{5}$?

- Rationalise the denominator of these fractions.

$$\frac{3+\sqrt{2}}{4-\sqrt{2}} \qquad \frac{2-\sqrt{3}}{1+\sqrt{3}} \qquad \frac{3-\sqrt{5}}{\sqrt{5}+5} \qquad \frac{1}{1-\sqrt{5}} \qquad \frac{a+\sqrt{b}}{c+\sqrt{b}}$$

Take it further

Two numbers are in the golden ratio if the ratio of the smaller to the larger is the same as the ratio of the larger to the sum of the two numbers.



So *a* and *b* are in the golden ratio if $a:b$ is the same as $b:a+b$.

The golden ratio written in surd form is $1:\dfrac{\sqrt{5}+1}{2}$ if the shorter length is 1, or $\dfrac{\sqrt{5}-1}{2}:1$ if the longer length is 1.

- Find the reciprocal of $\dfrac{\sqrt{5}+1}{2}$. Subtract 1 from $\dfrac{\sqrt{5}+1}{2}$. What do you notice?
- What are the two forms of the golden ratio as decimals?
- Find out more about the golden ratio.

Where this goes next

At A level rationalising the denominator is studied in Core Mathematics and applied in complex numbers which appear in Further Mathematics.

Teachers' notes

Task

- $(3-\sqrt{5})(3+\sqrt{5}) = 9-3\sqrt{5}+3\sqrt{5}-5 = 4$
- $(1+\sqrt{5})(3+\sqrt{5}) = 3+3\sqrt{5}+\sqrt{5}+5 = 8+4\sqrt{5}$
- $\dfrac{3+\sqrt{2}}{4-\sqrt{2}} = 1+\dfrac{\sqrt{2}}{2}$ $\quad \dfrac{2-\sqrt{3}}{1+\sqrt{3}} = \dfrac{3\sqrt{3}-5}{2}$ $\quad \dfrac{3-\sqrt{5}}{\sqrt{5}+5} = 1-\dfrac{2\sqrt{5}}{5}$ $\quad \dfrac{1}{1-\sqrt{5}} = \dfrac{-1-\sqrt{5}}{4}$

  $\dfrac{a+\sqrt{b}}{c+\sqrt{b}} = \dfrac{a+\sqrt{b}}{c+\sqrt{b}} \times \dfrac{c-\sqrt{b}}{c-\sqrt{b}} = \dfrac{ac-b+(c-a)\sqrt{b}}{c^2-b}$

Why do you multiply by $\dfrac{c-\sqrt{b}}{c-\sqrt{b}}$ to rationalise the denominator of $\dfrac{a+\sqrt{b}}{c+\sqrt{b}}$?

To explain this, you make use of an important connection to the fact that $(x+y)\times(x-y) = x^2-y^2$.

Putting c in place of x and $\sqrt{b}$ in place of y, you can see that $\left(c+\sqrt{b}\right)\times\left(c-\sqrt{b}\right) = c^2-b$ which is formed only of rational numbers.

Take it further

- The reciprocal of $\dfrac{\sqrt{5}+1}{2}$ and subtracting 1 from it both result in $\dfrac{\sqrt{5}-1}{2}$.

  Students could try using the diagram to explain this.
- The golden ratio as a decimal is 1.618 03… or 0.618 03… .
- http://goldennumber.net/ is a website devoted to the golden ratio and is probably a good place to start. *The Golden Ratio* by Mario Livio is an excellent book on the subject.