Trigonometry 1

What you should know

The trigonometric ratios: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

Pythagoras' theorem: opposite^2 + adjacent^2 = hypotenuse^2

New idea

If you divide $\sin\theta$ by $\cos\theta$ you get

$$\frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{\text{hyp}} \div \frac{\text{adj}}{\text{hyp}}$$
$$= \ldots$$
$$= \tan\theta$$

so $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$\sin^2\theta$ means $(\sin\theta)^2$ so you can write

$$\sin^2\theta = \left(\frac{\text{opp}}{\text{hyp}}\right)^2.$$

Using this idea you can add together $\sin^2\theta$ and $\cos^2\theta$ to get

$$\sin^2\theta + \cos^2\theta = \left(\frac{\text{opp}}{\text{hyp}}\right)^2 + \left(\frac{\text{adj}}{\text{hyp}}\right)^2$$
$$= \ldots$$
$$= 1$$

so $\sin^2\theta + \cos^2\theta = 1$.

Task: Trigonometric identities

- In both of the ideas above, what should the missing bits of algebra, represented by …, be? (There is more than one line of algebra missing!)
- Find the sine and cosine of some angles and check that, in every case, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$.
  Remember: $\sin^2\theta$ means $(\sin\theta)^2$ and $\cos^2\theta$ means $(\cos\theta)^2$.

Take it further

- You can use these identities to find values for $\cos\theta$ and $\tan\theta$ even if you only know $\sin\theta$. You don't even need to find out what the angle θ is.
  Try finding $\cos\theta$ and $\tan\theta$ when $\sin\theta = \frac{3}{5}$.
- Can you find some other right-angled triangles where $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all rational numbers (fractions that can be written as one integer over another)?
- Do these identities work when $\theta < 0°$ or $\theta > 90°$?
- Find out what **secant**, **cosecant** and **cotangent** are.
- Try dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$. What identity do you get?

Where this goes next

At A level trigonometry is studied in Core Mathematics.

Teachers' notes

Task

- $$\frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{\text{hyp}} \div \frac{\text{adj}}{\text{hyp}}$$

$$= \frac{\text{opp}}{\text{hyp}} \times \frac{\text{hyp}}{\text{adj}}$$

$$= \frac{\text{opp}}{\text{adj}}$$

$$= \tan\theta$$

$$\sin^2\theta + \cos^2\theta = \left(\frac{\text{opp}}{\text{hyp}}\right)^2 + \left(\frac{\text{adj}}{\text{hyp}}\right)^2$$

$$= \frac{\text{opp}^2}{\text{hyp}^2} + \frac{\text{adj}^2}{\text{hyp}^2}$$

$$= \frac{\text{opp}^2 + \text{adj}^2}{\text{hyp}^2}$$

$$= \frac{\text{hyp}^2}{\text{hyp}^2}$$

(by Pythagoras)

$$= 1$$

Take it further

- $$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\tfrac{3}{5}\right)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \tfrac{9}{25}$$

$$= \tfrac{16}{25}$$

$$\text{so} \quad \cos\theta = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5}$$

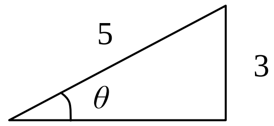
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \tfrac{3}{5} \div \tfrac{4}{5}$$

$$= \tfrac{3}{4}$$

- When $\sin\theta = \frac{3}{5}$, the triangle is based on the Pythagorean triple 3, 4, 5.
  For any right-angled triangle based on a Pythagorean triple $\sin\theta$, $\cos\theta$ and $\tan\theta$ will all be rational numbers.
  Other Pythagorean triples include 5, 12, 13 and 20, 21, 29.

- Here is one possible triangle for $\sin\theta = \frac{3}{5}$.
  The missing side length is 4 (by Pythagoras) and you can see that $\cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$.



When square rooting $\frac{16}{25}$ the value $\frac{-4}{5}$ could have been obtained.
This would correspond to an angle of approximately 143°.
The identities $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$ still work for angles less than 0° and greater than 90°.

- secant or $\sec\theta = \frac{1}{\cos\theta}$ or $\frac{\text{hyp}}{\text{adj}}$

  cosecant or $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ or $\frac{\text{hyp}}{\text{opp}}$

  cotangent or $\cot\theta = \frac{1}{\tan\theta}$ or $\frac{\text{adj}}{\text{opp}}$

- $$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Students can also divide through by $\sin^2\theta$ to get the identity

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \text{cosec}^2\theta$$